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ASYMPTOTIC HOMOGENIZATION OF VISCOELASTIC COMPOSITES WITH PERIODIC MICROSTRUCTURES

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Abstract—A systematic way of obtaining the effective viscoelastic moduli in time and frequency domain is presented for viscoelastic composites with periodic microstructures. The problem of estimating the effective moduli is formulated using the asymptotic homogenization method. For theoretical aspects, the memory effects due to the homogenization are shown in general form and a sufficient condition for the effects to disappear is fully discussed. The computational procedure is divided into two steps. The effective relaxation moduli are computed in Laplace transformed domain and are numerically inverse-transformed into time domain. The effective complex moduli are then readily obtained by using simple formulae of the Fourier transform. Several numerical examples are presented to illustrate and verify present approach and to discuss the memory effects. © 1998 Elsevier Science Ltd.

1. INTRODUCTION

Needs for estimating the effective moduli of composites have naturally arisen with their increasing engineering applications. Among many other methods of obtaining the effective moduli, the asymptotic homogenization method, which will be simply referred to homogenization method, has several attractive features mainly due to its systematic mathematical approach (Sanchez-Palencia, 1980; Bensoussan *et al.* 1978). Most importantly, the homogenization method guarantees convergence, i.e. the solution of the problem with a periodic microstructure converges to the solution of the homogenized problem as the period of the microstructure goes to zero. This implies that the accuracy of the homogenized solution is not significantly affected by the specific boundary conditions imposed on the global system. In fact, it has been shown that the homogenization method gives better estimates of the effective elastic moduli than standard mechanics approaches (Hollister and Kikuchi, 1992). Also, the homogenization method has nice features in a practical point of view. For the elastic case, the equations for the estimation of the effective elastic moduli have nearly the same form as those of elasticity problem. Thus the well-known numerical tools such as FEM can be easily implemented with slight modifications (Guedes and Kikuchi, 1990). This implies that the geometric configuration of microstructures do not matter. Also, the local fluctuations of field variables such as displacements can be easily computed from the homogenized solution and the estimation of the local errors due to homogenization is rather straightforward (Fish *et al.*, 1994).

Despite the attractive features of the homogenization method, there have been only a few works on the applications to viscoelasticity on the contrary to the elastic cases. The effective complex moduli at fixed frequencies are computed based on the homogenization method using the correspondence principle (Nguyen *et al.*, 1995). In that case, the homogenization process is identical to the one used in the elastic case. The discussions on the estimation of the effective relaxation moduli or the effective creep compliances of general viscoelastic composites in time domain are rarely found although works on the simple Voigt and Maxwell models of viscoelasticity are available (Sanchez-Palencia, 1980; Suquet,

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1987). They showed that the memory effects appeared due to the homogenization. Although the Voigt and the Maxwell models have only instantaneous memories, long-term memories were induced during the homogenization process. From the mathematical studies of differential equations with one space and time dimension, it has been found that the memory effects appear due to the interaction between the spatial and the temporal variations of the coefficients of the differential equations (Tartar, 1990).

In the present work, the homogenization process is formulated for general viscoelastic composites and a systematic way of obtaining the effective relaxation moduli in time domain is presented. Also, the memory effects are shown in general form and are discussed in detail. It is shown with several numerical examples that the appearance of the memory effects makes the computations very complicated. It is also shown that the memory effects may be used in a positive manner to obtain the required damping characteristics of viscoelastic composites.

2. HOMOGENIZATION IN VISCOELASTICITY

2.1. *Introduction*

The homogenization method has been developed from the studies of partial differential equations with rapidly oscillating coefficients. The method has been applied to the estimation of the effective moduli of composites with periodic microstructures (Sanchez-Palencia, 1980). A heterogeneous medium with a periodic structure may be replaced by an effective homogeneous one, provided that the period is very small compared to the global length scale. In the homogenization method, the field variables are assumed to vary in multiple scales, i.e. in local and global scales, and thus they are represented by asymptotic expansions in each spatial scale. Due to the periodicity of the microstructure, field variables such as displacement, strain and stress are assumed to be periodic with respect to the local scale. The effective moduli are then obtained by investigating the asymptotic behavior of the medium as the period of the microstructure approaches to zero.

2.2. *Basic concepts for homogenization*

Before the formulation, several basic concepts and notations need to be defined. In order to deal with two different length scales associated with microscopic and macroscopic behaviors, global coordinate is designated by $\mathbf{x}$ and local coordinate by $\mathbf{y}$. The global coordinate and the local coordinate are related with each other by a positive real parameter ε as follows:

$$\mathbf{y} = \mathbf{x}/\varepsilon. \tag{1}$$

Y-periodicity of a function $f(\mathbf{y})$ in the local coordinate is defined as follows:

$$f(y_1+n_1Y_1, y_2+n_2Y_2, y_3+n_3Y_3) = f(y_1, y_2, y_3),$$
$$\forall \mathbf{y} = (y_1, y_2, y_3) \in R^3 \quad \text{and} \quad \forall \text{ integers } n_1, n_2 \text{ and } n_3 \tag{2}$$

where Y_1, Y_2 and Y_3 represent the period of the Y-periodicity, i.e.

$$Y = (0, Y_1) \times (0, Y_2) \times (0, Y_3). \tag{3}$$

From a Y-periodic function $f(\mathbf{y})$ in the local coordinate, an εY-periodic function $f^{\varepsilon}(\mathbf{x})$ in the global coordinate can be defined as follows:

$$f^{\varepsilon}(\mathbf{x}) = f(\mathbf{x}/\varepsilon) = f(\mathbf{y}). \tag{4}$$

Derivatives in the global scale and those in the local scale can be related using eqn (1). Suppose that a function $g^{\varepsilon}(\mathbf{x}) = g(\mathbf{x}, \mathbf{y})$ depends on both the global and the local coordinates. Then the following relations hold:

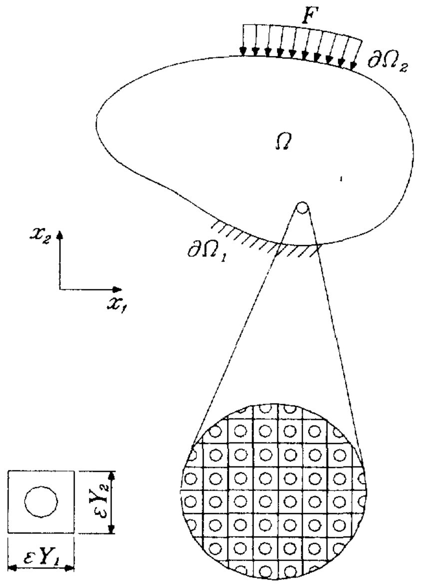


Fig. 1. A problem with periodic microstructure.

$$\frac{\partial g^{\varepsilon}(\mathbf{x})}{\partial x_i} = \frac{\partial g(\mathbf{x},\mathbf{y})}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial g(\mathbf{x},\mathbf{y})}{\partial y_i}; \quad \mathbf{y} = \mathbf{x}/\varepsilon. \tag{5}$$

For the averaging process of the homogenization, the following mean operator, $\tilde{}$, on Y is defined.

$$\tilde{\bullet} = \frac{1}{|Y|}\int_Y \bullet \, \mathrm{d}y \tag{6}$$

where $|Y|$ is the measure of Y.

2.3. *Problem statements*

The following problem of viscoelasticity with a periodic microstructure as shown in Fig. 1 is considered, where the inertia effects and the body forces are not present.

$$\frac{\partial \sigma^{\varepsilon}_{ij}(\mathbf{x},t)}{\partial x_j} = 0 \quad \text{in} \quad \Omega \tag{7}$$

$$u^{\varepsilon}_i(\mathbf{x},t) = 0 \quad \text{on} \quad \partial_1\Omega \tag{8}$$

$$\sigma^{\varepsilon}_{ij}(\mathbf{x},t)n_j = F_i(\mathbf{x},t) \quad \text{on} \quad \partial_2\Omega \tag{9}$$

$$u^{\varepsilon}_i(\mathbf{x},t) = 0 \quad \text{and} \quad \frac{\partial u^{\varepsilon}_i(\mathbf{x},t)}{\partial t} = 0 \quad \text{at} \quad t = 0 \tag{10}$$

$$\begin{aligned}\sigma^{\varepsilon}_{ij}(\mathbf{x},t) &= G^{\varepsilon}_{ijkl}(\mathbf{x},t)e_{kl}(\mathbf{u}^{\varepsilon}(\mathbf{x},0)) + \int_0^t G^{\varepsilon}_{ijkl}(\mathbf{x},t-\tau)\frac{\partial e_{kl}(\mathbf{u}^{\varepsilon}(\mathbf{x},\tau))}{\partial \tau}\,\mathrm{d}\tau \\ &= \int_0^t G^{\varepsilon}_{ijkl}(\mathbf{x},t-\tau)\frac{\partial e_{kl}(\mathbf{u}^{\varepsilon}(\mathbf{x},\tau))}{\partial \tau}\,\mathrm{d}\tau\end{aligned} \tag{11}$$

$$e_{kl}(\mathbf{u}^{\varepsilon}(\mathbf{x}, t)) = \frac{1}{2}\left(\frac{\partial u_k^{\varepsilon}(\mathbf{x}, t)}{\partial x_l} + \frac{\partial u_l^{\varepsilon}(\mathbf{x}, t)}{\partial x_k}\right) \tag{12}$$

where Ω is an open connected domain of R^3, $\partial_1\Omega$ the displacement-prescribed boundary, $\partial_2\Omega$ the traction-prescribed boundary, and F_i the traction force on $\partial_2\Omega$. The relaxation tensor $G_{ijkl}^{\varepsilon}(\mathbf{x}, t)$ is given by

$$G_{ijkl}^{\varepsilon}(\mathbf{x}, t) = G_{ijkl}(\mathbf{x}/\varepsilon, t) = G_{ijkl}(\mathbf{y}, t) \tag{13}$$

where $G_{ijkl}(\mathbf{y}, t)$ is Y-periodic in the local coordinate $\mathbf{y}$ and satisfy the symmetry and the positivity conditions:

$$G_{ijkl}(\mathbf{y}, t) = G_{jikl}(\mathbf{y}, t) = G_{ijlk}(\mathbf{y}, t) = G_{klij}(\mathbf{y}, t), \quad \forall \mathbf{y} \in Y \tag{14}$$

$$\exists \alpha > 0 \text{ such that } G_{ijkl}(\mathbf{y}, t)e_{ij}e_{kl} > \alpha e_{ij}e_{ij}, \quad \forall \text{ symmetric } e_{ij} \text{ and } \forall \mathbf{y} \in Y. \tag{15}$$

Since $G_{ijkl}^{\varepsilon}(\mathbf{x}, t)$ is dependent on ε, $u_i^{\varepsilon}(\mathbf{x}, t)$ is also dependent on ε. The asymptotic behavior is to be observed as ε approaches to zero.

2.4. *Asymptotic expansions*

The most important and essential postulate in the homogenization method is that $u_i^{\varepsilon}(\mathbf{x}, t)$ has an asymptotic expansion in the following form.

$$u_i^{\varepsilon}(\mathbf{x}, t) = u_i^0(\mathbf{x}, t) + \varepsilon u_i^1(\mathbf{x}, \mathbf{y}, t) + \varepsilon^2 u_i^2(\mathbf{x}, \mathbf{y}, t) + , \ldots, ; \quad \mathbf{y} = \mathbf{x}/\varepsilon \tag{16}$$

where $u_i^m(\mathbf{x}, \mathbf{y}, t)$ is Y-periodic in $\mathbf{y}$ and independent of ε. From eqns (5) and (16), the asymptotic expansions for e_{ij}^{ε} and $\sigma_{ij}^{\varepsilon}$ are obtained as follows:

$$e_{ij}^{\varepsilon}(\mathbf{x}, t) = e_{ij}(\mathbf{u}^{\varepsilon}(\mathbf{x}, t)) = e_{ij}^0(\mathbf{x}, \mathbf{y}, t) + \varepsilon e_{ij}^1(\mathbf{x}, \mathbf{y}, t) + , \ldots, ; \quad \mathbf{y} = \mathbf{x}/\varepsilon \tag{17}$$

$$\begin{aligned} \sigma_{ij}^{\varepsilon}(\mathbf{x}, t) &= \int_0^t G_{ijkl}(\mathbf{y}, t-\tau)\frac{\partial e_{kl}^{\varepsilon}(\mathbf{x}, \tau)}{\partial \tau}\,\mathrm{d}\tau \\ &= \sigma_{ij}^0(\mathbf{x}, \mathbf{y}, t) + \varepsilon\sigma_{ij}^1(\mathbf{x}, \mathbf{y}, t) + , \ldots, ; \quad \mathbf{y} = \mathbf{x}/\varepsilon \end{aligned} \tag{18}$$

where

$$e_{ij}^0(\mathbf{x}, \mathbf{y}, t) = e_{ijx}(\mathbf{u}^0) + e_{ijy}(\mathbf{u}^1) \tag{19a}$$

$$\sigma_{ij}^0(\mathbf{x}, \mathbf{y}, t) = \int_0^t G_{ijkl}(\mathbf{y}, t-\tau)\frac{\partial e_{kl}^0(\mathbf{x}, \mathbf{y}, \tau)}{\partial \tau}\,\mathrm{d}\tau. \tag{19b}$$

In eqn (19) the subscripts x and y mean the differentiation with respect to x_i and y_i, respectively, i.e.

$$e_{ijx}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \quad \text{and} \quad e_{ijy}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i}\right). \tag{20}$$

2.5. *Local problem, global problem and homogenized moduli*

By introducing eqns (5) and (18) into eqn (7), and arranging it against ε^{-1} and ε^0, we obtain the following two equations.

$$\frac{\partial \sigma^0_{ij}(\mathbf{x}, \mathbf{y}, t)}{\partial y_j} = 0 \tag{21}$$

$$\frac{\partial \sigma^1_{ij}(\mathbf{x}, \mathbf{y}, t)}{\partial y_j} + \frac{\partial \sigma^0_{ij}(\mathbf{x}, \mathbf{y}, t)}{\partial x_j} = 0. \tag{22}$$

Noting that $\sigma^1_{ij}(\mathbf{x}, \mathbf{y}, t)$ is Y-periodic in $\mathbf{y}$, we obtain

$$\frac{1}{|Y|}\int_Y \frac{\partial \sigma^1_{ij}}{\partial y_j}\,\mathrm{d}y = \frac{1}{|Y|}\int_{\partial Y} \sigma^1_{ij} n_j\,\mathrm{d}S = 0 \quad \text{as} \quad \varepsilon \to 0. \tag{23}$$

Thus, by applying the mean operator, eqn (22) becomes

$$\frac{\partial \tilde{\sigma}^0_{ij}(\mathbf{x}, t)}{\partial x_j} = 0. \tag{24}$$

Equations (21) and (24) represent the local and the global problems, respectively. By solving the local problem (21), homogenized viscoelastic moduli can be obtained. From eqn (19), we obtain

$$\sigma^0_{ij}(\mathbf{x}, \mathbf{y}, t) = \int_0^t G_{ijkl}(\mathbf{y}, t-\tau)\frac{\partial}{\partial \tau}[e_{klx}(\mathbf{u}^0(\mathbf{x}, \tau)) + e_{kly}(\mathbf{u}^1(\mathbf{x}, \mathbf{y}, \tau))]\,\mathrm{d}\tau. \tag{25}$$

Plugging eqn (25) into eqn (21) and applying Laplace transform, the local problem becomes

$$-\frac{\partial}{\partial y_j}[s\hat{G}_{ijkl}(\mathbf{y}, s) e_{kly}(\hat{\mathbf{u}}^1(\mathbf{x}, \mathbf{y}, s))] = e_{klx}(\hat{\mathbf{u}}^0(\mathbf{x}, s))\frac{\partial (s\hat{G}_{ijkl}(\mathbf{y}, s))}{\partial y_j} \tag{26}$$

where variables with ^ show that they are Laplace transformed. For the variational formulation of eqn (26), we introduce Hilbert spaces with the association inner products as follows:

$$H_Y = \{\mathbf{u} \mid u_i \in L^2_{\mathrm{loc}}(R^3), \quad Y\text{-periodic}\} \tag{27a}$$

$$V_Y = \{\mathbf{u} \mid u_i \in H^1_{\mathrm{loc}}(R^3), \quad Y\text{-periodic}\} \tag{27b}$$

$$(\mathbf{u}, \mathbf{v})_{H_Y} = \int_Y u_i v_i\,\mathrm{d}y \tag{28a}$$

$$(\mathbf{u}, \mathbf{v})_{V_Y} = (\mathbf{u}, \mathbf{v})_{H_Y} + \sum_{i=1}^{3}\left(\frac{\partial \mathbf{u}}{\partial y_i}, \frac{\partial \mathbf{v}}{\partial y_i}\right)_{H_Y}. \tag{28b}$$

Using periodicity condition, following variational formulation of the local problem is obtained in Laplace transformed domain.

Find $\hat{\mathbf{u}}^1(\mathbf{x}, \mathbf{y}, s)$ with values in V_Y such that $\forall \mathbf{v} \in V_Y$,

$$\int_Y s\hat{G}_{ijkl}(\mathbf{y}, s) e_{kly}(\hat{\mathbf{u}}^1(\mathbf{x}, \mathbf{y}, s)) e_{ijy}(\mathbf{v})\,\mathrm{d}y = -e_{klx}(\hat{\mathbf{u}}^0(\mathbf{x}, \mathbf{y}, s))\int_Y s\hat{G}_{ijkl}(\mathbf{y}, s) e_{ijy}(\mathbf{v})\,\mathrm{d}y. \tag{29}$$

In eqn (29), supposing that $\hat{\mathbf{u}}^0$ is given, $\hat{\mathbf{u}}^1$ can be obtained in terms of $\hat{\mathbf{u}}^0$. However, the problem (29) does not have a unique solution, but only can determine the solution up to an additive constant which depends only on **x** (Sanchez-Palencia, 1980). To treat this problem, we define a space with an associated inner product as follows:

$$\tilde{V}_Y = \{\mathbf{u} \in V_Y \mid \tilde{\mathbf{u}} = \mathbf{0}\} \tag{30}$$

$$(\mathbf{u}, \mathbf{v})_{\tilde{V}_Y} = \int_Y s\hat{G}_{ijkl}(\mathbf{y}, s) e_{kly}(\mathbf{u}) e_{ijy}(\mathbf{v}) \, \mathrm{d}y. \tag{31}$$

Let us also define $\hat{\mathbf{w}}_{kl}(\mathbf{y}, s)$ by

$$(\hat{\mathbf{w}}_{kl}(\mathbf{y}, s), \mathbf{v})_{\tilde{V}_Y} = \int_Y s\hat{G}_{ijkl}(\mathbf{y}, s) e_{ijy}(\mathbf{v}) \, \mathrm{d}y, \quad \forall \mathbf{v} \in \tilde{V}_Y. \tag{32}$$

Then eqn (29) becomes

$$(e_{klx}(\hat{\mathbf{u}}^0(\mathbf{x}, s)) \hat{\mathbf{w}}_{kl}(\mathbf{y}, s) + \hat{\mathbf{u}}^1(\mathbf{x}, \mathbf{y}, s), \mathbf{v})_{\tilde{V}_Y} = 0, \quad \forall \mathbf{v} \in \tilde{V}_Y. \tag{33}$$

Therefore, $\hat{\mathbf{u}}^1$ is given in terms of $\hat{\mathbf{u}}^0$ and $\hat{\mathbf{w}}_{kl}$ as follows:

$$\hat{\mathbf{u}}^1(\mathbf{x}, \mathbf{y}, s) = -e_{klx}(\hat{\mathbf{u}}^0(\mathbf{x}, s)) \hat{\mathbf{w}}_{kl}(\mathbf{y}, s) + \mathbf{C}(\mathbf{x}, s). \tag{34}$$

By applying inverse Laplace transform to eqn (34), and plugging it into eqn (25), we obtain

$$\sigma^0_{ij}(\mathbf{x}, \mathbf{y}, t) = \int_0^t G_{ijkl}(\mathbf{y}, t-\tau) \frac{\partial}{\partial \tau} \left[e_{klx}(\mathbf{u}^0(\mathbf{x}, \tau)) - \int_0^\tau e_{mnx}(\mathbf{u}^0(\mathbf{x}, \tau)(\tau - p)) e_{kly}(\mathbf{w}_{mn}(\mathbf{y}, \tau)(p)) \, \mathrm{d}p \right] \mathrm{d}\tau \tag{35}$$

where $\mathbf{w}_{mn}$ is the inverse Laplace transform of $\hat{\mathbf{w}}_{mn}$. By using the Leibnitz rule and by changing the order of the integrations, we obtain

$$\sigma^0_{ij}(\mathbf{x}, \mathbf{y}, t) = \int_0^t \left[G_{ijkl}(\mathbf{y}, t-\tau) - \int_0^{t-\tau} G_{ijmn}(\mathbf{y}, (t-\tau) - p) e_{mny}(\mathbf{w}_{kl}(\mathbf{y}, p)) \, \mathrm{d}p \right] \times \frac{\partial}{\partial \tau} e_{klx}(\mathbf{u}^0(\mathbf{x}, \tau)) \, \mathrm{d}\tau. \tag{36}$$

By applying the mean operator, the following homogenized stress–strain relations are obtained from eqn (36),

$$\tilde{\sigma}^0_{ij}(\mathbf{x}, t) = \int_0^t G^h_{ijkl}(t-\tau) \frac{\partial}{\partial \tau} e_{klx}(\mathbf{u}^0(\mathbf{x}, \tau)) \, \mathrm{d}\tau \tag{37}$$

where homogenized viscoelastic relaxation modulus, $G^h_{ijkl}(t)$, is given by the following equation.

$$G^h_{ijkl}(t) = \left[G_{ijkl}(\mathbf{y}, t) - \int_0^t G_{ijmn}(\mathbf{y}, t-p) e_{mny}(\mathbf{w}_{kl}(\mathbf{y}, p)) \, \mathrm{d}p \right]^{\sim}. \tag{38}$$

It can be easily shown that $G^h_{ijkl}(t)$ satisfies the symmetry and the positivity conditions.

Consequently, the homogenized problem with the homogenized viscoelastic moduli can be stated as follows:

$$\frac{\partial \bar{\sigma}_{ij}^{0}(\mathbf{x},t)}{\partial x_j} = 0 \quad \text{in} \quad \Omega \tag{39}$$

$$u_i^0(\mathbf{x},t) = 0 \quad \text{on} \quad \partial_1\Omega \tag{40}$$

$$\bar{\sigma}_{ij}^{0}(\mathbf{x},t)n_j = F_i(\mathbf{x},t) \quad \text{on} \quad \partial_2\Omega \tag{41}$$

$$u_i^0(\mathbf{x},t) = 0 \quad \text{and} \quad \frac{\partial u_i^0(\mathbf{x},t)}{\partial t} = 0 \quad \text{at} \quad t = 0 \tag{42}$$

$$\bar{\sigma}_{ij}^{0}(\mathbf{x},t) = \int_0^t G_{ijkl}^h(t-\tau)\frac{\partial e_{kl}(\mathbf{u}^0(\mathbf{x},\tau))}{\partial \tau}\,\mathrm{d}\tau \tag{43}$$

$$e_{kl}(\mathbf{u}^0(\mathbf{x},t)) = \frac{1}{2}\left(\frac{\partial u_k^0(\mathbf{x},t)}{\partial x_l} + \frac{\partial u_l^0(\mathbf{x},t)}{\partial x_k}\right). \tag{44}$$

Note that the homogenized equations [eqns (39)–(44)] are in exactly the same form as the original ones with microstructure [eqns (7)–(12)] if the original modulus, $G_{ijkl}^{\varepsilon}(\mathbf{x},t)$, is replaced by the corresponding homogenized modulus, $G_{ijkl}^h(t)$. Thus complicated problems with microstructures can be replaced by corresponding simple problems without microstructures. However, the homogenized modulus, $G_{ijkl}^h(t)$, which is in a sense the averaged modulus of the material with the microstructure, needs to be calculated beforehand.

2.6. *Computation of the homogenized relaxation moduli*

By taking Laplace transform of eqn (38), we obtain

$$s\hat{G}_{ijkl}^h(s) = \{s\hat{G}_{ijmn}(\mathbf{y},s)[\delta_{mk}\,\delta_{nl} - e_{mny}(\hat{\mathbf{w}}_{kl}(\mathbf{y},s))]\}^{\tilde{}} \tag{45}$$

where $\hat{\mathbf{w}}_{kl}(\mathbf{y},s)$ is the solution of the following local problem in Laplace transformed domain.

$$\int_Y s\hat{G}_{ijmn}(\mathbf{y},s)e_{mny}(\hat{\mathbf{w}}_{kl}(\mathbf{y},s))e_{ijy}(\mathbf{v})\,\mathrm{d}y = \int_Y s\hat{G}_{ijkl}(\mathbf{y},s)e_{ijy}(\mathbf{v})\,\mathrm{d}y, \quad \forall \mathbf{v} \in \tilde{V}_Y. \tag{46}$$

Thus, the homogenized modulus, $G_{ijkl}^h(t)$, can be obtained as follows. First, $\hat{\mathbf{w}}_{kl}(\mathbf{y},s)$ is computed in Laplace transformed domain from eqn (46). Note that eqn (46), from which $\hat{\mathbf{w}}_{kl}(\mathbf{y},s)$ is computed, has nearly the same form as the typical elastostatic problems. The domain of the local problem is the unit cell, Y. The right-hand side of eqn (46) acts like body forces and the boundary conditions are given as the periodic boundary conditions on $\hat{\mathbf{w}}_{kl}(\mathbf{y},s)$. This means that $\hat{\mathbf{w}}_{kl}(\mathbf{y},s)$ can be easily computed by the already existing FEM codes with slight modifications. After $\hat{\mathbf{w}}_{kl}(\mathbf{y},s)$ is computed, the homogenized modulus in Laplace transformed domain, $\hat{G}_{ijkl}^h(s)$, can be readily obtained from eqn (45). Then $G_{ijkl}^h(t)$ is obtained from $\hat{G}_{ijkl}^h(s)$ by inverse Laplace transform. Detailed procedure for numerical inverse Laplace transform is to be explained in the next section. It should be mentioned that the homogenization in Laplace transformed domain, eqns (45) and (46), could also be obtained by using the correspondence principle and the homogenization formulation in elasticity (Sanchez-Palencia, 1980). Thus the correspondence principle implies that the homogenizations both in time domain and in Laplace transformed domain are equivalent.

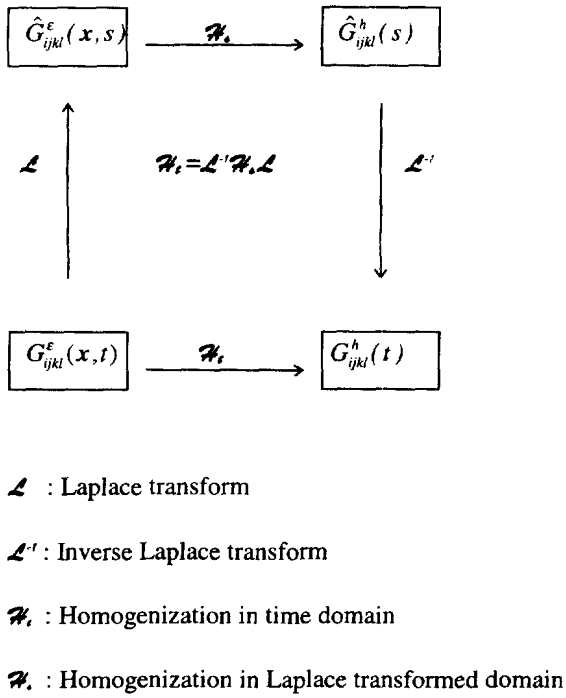


Fig. 2. Schematic diagram of relations between moduli in transformed domains.

This is schematically shown in Fig. 2. Note that the homogenization operators $\mathscr{H}_t$ and $\mathscr{H}_L$ are not linear operators.

3. MEMORY EFFECTS AND INVERSE LAPLACE TRANSFORM

3.1. *Introduction*

Viscoelastic materials have memories in the sense that the current behaviors are dependent on the past histories of stresses and strains. In addition to the original memories of the constituent materials of viscoelastic composites, the homogenization process induces additional long-term memories. This behavior is termed as the memory effects. It has been shown that the long-term memory appears as a result of homogenization for composites in which the viscoelastic phase is treated with a Voigt model which only has the instantaneous memory (Sanchez-Palencia, 1980). The similar result has been obtained for a Maxwell model (Suquet, 1987). In the following, the memory effects are discussed in general form.

3.2. *Memory effects*

The integral term in eqn (38) represents the memory effects. It can be easily seen that the homogenized relaxation modulus depend on the history of the relaxation moduli of the constituent materials as well as on the current moduli of them. However, the memory effects do not appear in every situation. If the relaxation modulus, $G_{ijkl}(\mathbf{y}, t)$, is separable with respect to space and time, the memory effects do not appear. This fact can be shown as follows. Suppose that $G_{ijkl}(\mathbf{y}, t)$ is separable in $\mathbf{y}$ and t, i.e.

$$G_{ijkl}(\mathbf{y}, t) = F_{ijkl}(\mathbf{y})T(t). \tag{47}$$

Then the Laplace transform of $G_{ijkl}(\mathbf{y}, t)$ is given by

$$s\hat{G}_{ijkl}(\mathbf{y}, s) = sF_{ijkl}(\mathbf{y})\hat{T}(s). \tag{48}$$

Plugging eqn (48) into eqn (46), the following local problem is obtained.

$$s\hat{T}(s)\int_Y F_{ijmn}(\mathbf{y})e_{mny}(\hat{\mathbf{w}}_{kl})e_{ijy}(\mathbf{v})\,\mathrm{d}y = s\hat{T}(s)\int_Y F_{ijkl}(\mathbf{y})e_{ijy}(\mathbf{v})\,\mathrm{d}y, \quad \forall\mathbf{v}\in\tilde{V}_Y. \tag{49}$$

From eqn (49), it is evident that $\hat{\mathbf{w}}_{kl}$ is not a function of s. Then from eqn (45), we obtain

$$\begin{aligned} s\hat{G}^h_{ijkl}(s) &= \widetilde{\{s\hat{T}(s)F_{ijmn}(\mathbf{y})[\delta_{mk}\,\delta_{nl}-e_{mny}(\hat{\mathbf{w}}_{kl}(\mathbf{y}))]\}} \\ &= s\hat{T}(s)\widetilde{\{F_{ijmn}(\mathbf{y})[\delta_{mk}\,\delta_{nl}-e_{mny}(\hat{\mathbf{w}}_{kl}(\mathbf{y}))]\}} \\ &= s\hat{T}(s)F^h_{ijkl}. \end{aligned} \tag{50}$$

The inverse Laplace transform of eqn (50) gives

$$G^h_{ijkl}(t) = F^h_{ijkl}T(t). \tag{51}$$

From the above result, it can be seen that the memory effects come from the coupling effects of the spatial and the temporal variations of the viscoelastic modulus. It has been discussed that the memory effects may be induced during the homogenization process by oscillations in spatial properties for the partial differential equations in which both space and time are involved (Tartar, 1990).

Suppose that the relaxation moduli are represented in the Prony series as follows:

$$\mathbf{G}(\mathbf{y},t) = \mathbf{G}_\infty(\mathbf{y})+\sum_{i=1}^{n}\mathbf{G}_i(\mathbf{y})\,\mathrm{e}^{-t/\tau_i}. \tag{52}$$

If space variable and time variable are separable, then the following holds:

$$\mathbf{G}_i(\mathbf{y}) = \alpha_i\mathbf{G}_\infty(\mathbf{y}) \text{ for some scalars } \alpha_i, \quad i = 1,\ldots,n. \tag{53}$$

For example, for an isotropic material with voids which has a constant Poisson's ratio, the memory effects do not appear. For a composite material with the viscoelastic matrix and the elastic fibers, the memory effects appear. However, if the elastic fibers are much stiffer than the viscoelastic matrix, the memory effects become negligible.

The memory effects make the homogenization problem complicated. When the relaxation moduli are separable as can be seen in eqn (51), it is sufficient to solve the local problem only once. However, it is necessary to solve the local problem for every s if the relaxation moduli are not separable with respect to space and time variables. Thus, in practice, the effective relaxation moduli should be curve-fitted after computing them for several s. This fact brings in a lot of complications as discussed in the following section.

3.3. *Inverse Laplace transform*

Since homogenized relaxation moduli are computed in Laplace transformed domain, inverse Laplace transform is required unless the memory effects do not appear. In the present work, the least-square fitting in Laplace transformed domain is employed based on the Prony series representations of the relaxation moduli. Since it is difficult to optimize the relaxation times of the fitting function because of their nonlinear behavior, the relaxation times should be properly chosen in the region where the relaxation curve changes rapidly (Cost and Becker, 1970). The choice of the fitting function, i.e the number of terms and the values of the relaxation times, is carried out as follows. Firstly, two bounding points between which the relaxation curve changes rapidly are selected. Secondly, at least one point per decade is selected as a relaxation time with even spacing in the log scale. Of course, the number of data points should be more than or equal to the number of fitting points. In general, because of memory effects, more terms in the Prony series are required in the fitting function than the number of terms in the Prony series representations of the given material. Once the approximations are made in the Laplace transformed domain, the

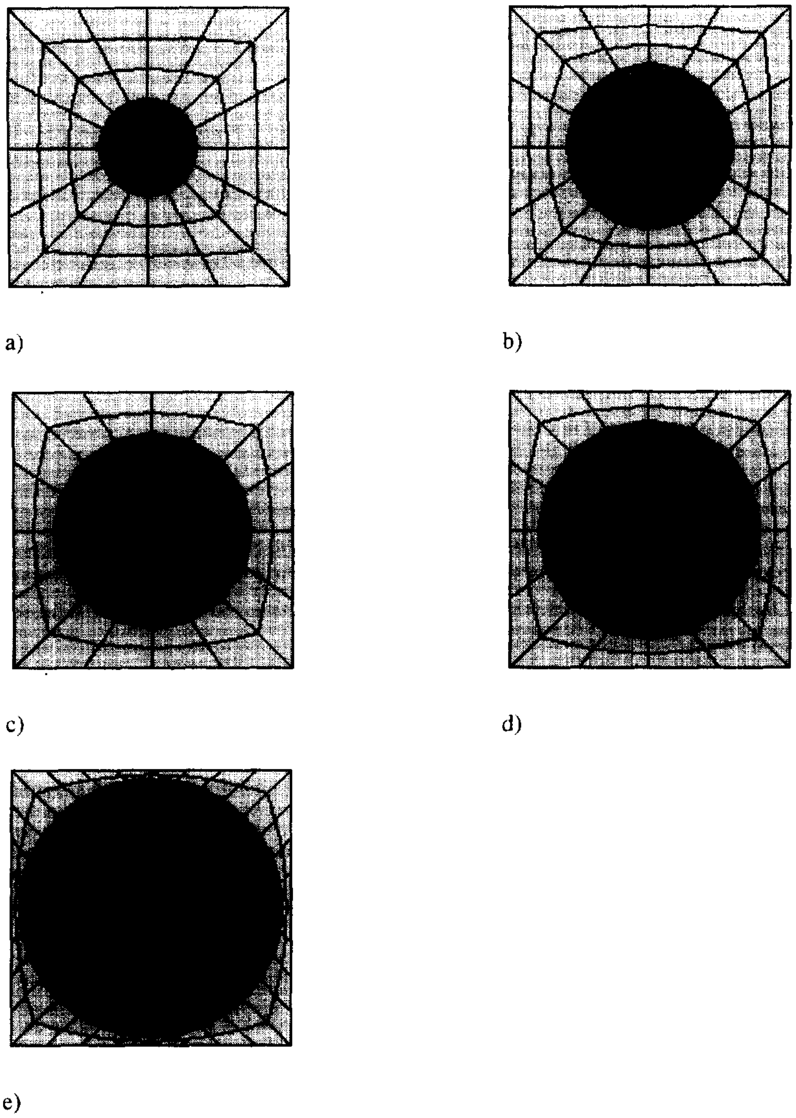


Fig. 3. Finite element meshes of microstructures: (a) vf = 10%; (b) vf = 30%; (c) vf = 40%; (d) vf = 50%; (e) vf = 70%.

relaxation moduli in time domain and the complex moduli (the storage moduli and the loss moduli) as well as the loss tangent in frequency domain are readily obtained using the known relations (Christensen, 1982).

4. NUMERICAL EXAMPLES AND DISCUSSIONS

In the following examples, two-dimensional plane stress state is considered and constituent materials are assumed isotropic. All the microstructures used in the examples have the same configuration, i.e. a circular inclusion in a square matrix. The size of the inclusion and the material properties of matrix and inclusion are to be varied. Volume fractions of the inclusions are 10, 30, 40, 50 and 70%. Finite element meshes for these microstructures are shown in Fig. 3. It should be mentioned that excessively fine meshes are not required for the present microstructures due to their simple shapes. We had refined the meshes for all cases and had examined the differences in the homogenized moduli. Even when the number of elements were increased about ten times, the differences were in about 2% or less. The meshes in Fig. 3 are sufficiently fine enough in our case and will be used in the following examples.

4.1. *Example* 1

In order to observe the memory effects, a viscoelastic composite with elastic circular inclusions in viscoelastic matrix in which the stiffness of the elastic inclusions is comparable to that of the viscoelastic matrix is treated. The modulus of the elastic inclusions is

$$E = 20, \quad \nu = 0.21. \tag{54}$$

The modulus of the viscoelastic matrix, which is represented by using the standard linear solid model (Christensen, 1982), is given by

$$E(t) = 3+17\,e^{-t}, \quad \nu = 0.38. \tag{55}$$

The volume fraction of the inclusions is 40%. Because of the long-term memory effects, the homogenized relaxation modulus cannot be represented exactly with the standard linear solid model. It is, therefore, required to introduce additional terms in Prony series approximation. To see the effect of the number of the terms in Prony series approximation, the numbers of relaxation terms (N_f) is varied by 1, 3, 5 and 7. Figure 4 shows the results. They show that small errors introduced in fitting the homogenized relaxation modulus in Laplace transformed domain may cause large errors in the computed complex modulus and the loss tangent in frequency domain. Therefore, the proper choice of the number of decades and the values of the relaxation times for fitting the homogenized relaxation modulus becomes important when the memory effects become apparent. In the present case, it may be concluded that at least five relaxation times in the interval, where the relaxation curves vary rapidly, should be used to accurately fit the homogenized relaxation modulus.

Next, the homogenized modulus was computed by varying the volume fractions of the inclusions from 10 to 70%. Figure 5 shows the results. The amount of changes in the homogenized relaxation modulus as a function of s or t decreases monotonically as the volume fraction of the inclusions increase. Also, the loss tangent decreases monotonically as the volume fraction increases. However, the relaxation modulus itself does not monotonically decrease or increase as a function of the volume fraction since the Poisson's ratio of the inclusion and that of the matrix are different. The frequency of peak loss tangent varies according to the volume fraction. It should be noted that higher loss tangent may be obtained, even when the volume fraction of the inclusion is increased, by changing the configuration of the microstructure even if the volume fraction of the inclusion is increased.

4.2. *Example* 2

As a second example, the following viscoelastic material with one additional term in the Prony series is used as the viscoelastic matrix of the composite.

$$E(t) = 3+7\,e^{-t}+10\,e^{-t/10}, \quad \nu = 0.38. \tag{56}$$

Figures 6 and 7 show the results. They show that the previous discussions apply more vividly for the present case. Due to the memory effect, the proper choice of the fitting function is also very important in the estimation of the effective moduli.

4.3. *Example* 3

Composites with the microstructure composed of two isotropic viscoelastic materials with different relaxation times are used. The relaxation moduli are given as follows:

$$E(t) = 3+17\,e^{-t/10}, \quad \nu = 0.38 \quad \text{for material 1} \tag{57}$$

$$E(t) = 3+17\,e^{-t}, \quad \nu = 0.38 \quad \text{for material 2.} \tag{58}$$

The effective moduli of the two composites are estimated. In the first composite, material 1 is for the circular inclusion and material 2 for the matrix, and in the second one, the roles

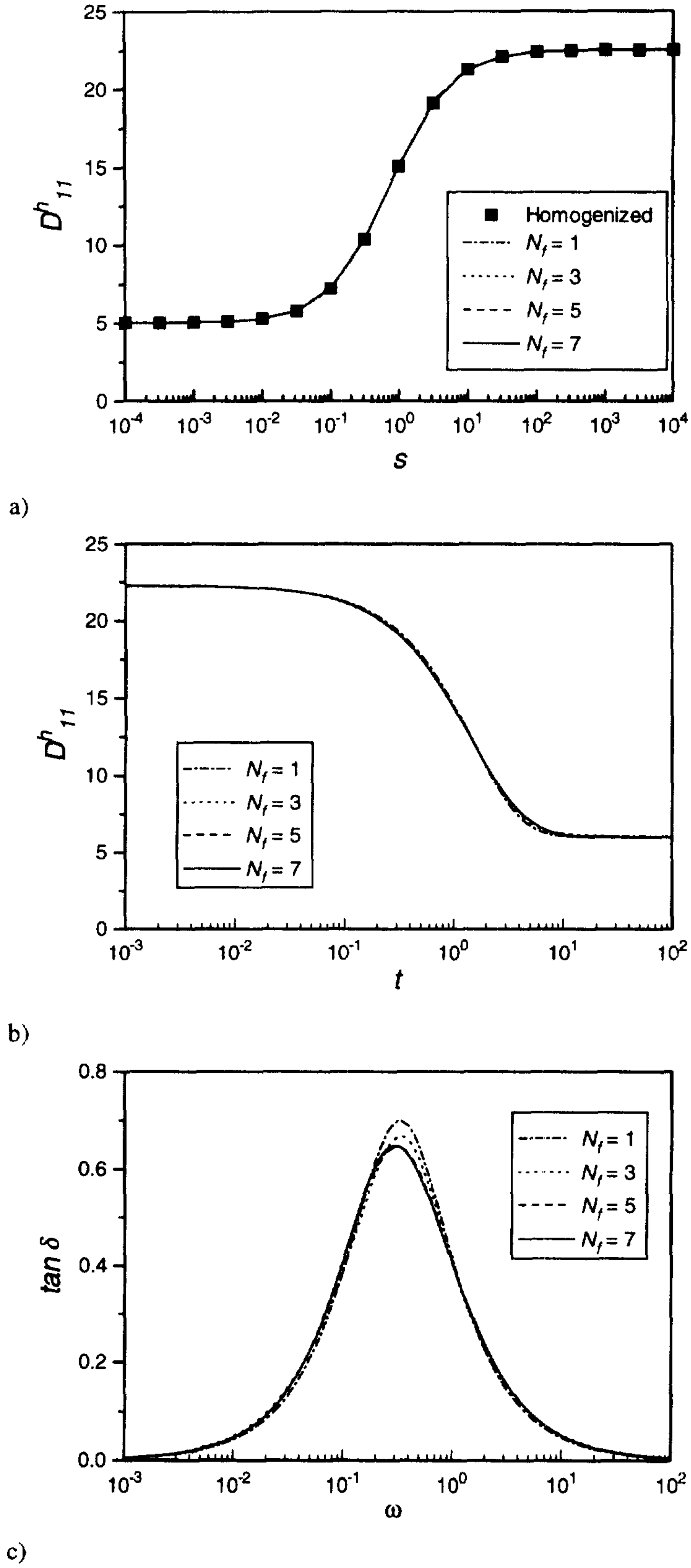


Fig. 4. Effect of number of fitting terms: (a) effective modulus in Laplace transformed domain; (b) effective modulus in time domain; and (c) effective loss tangent in frequency domain.

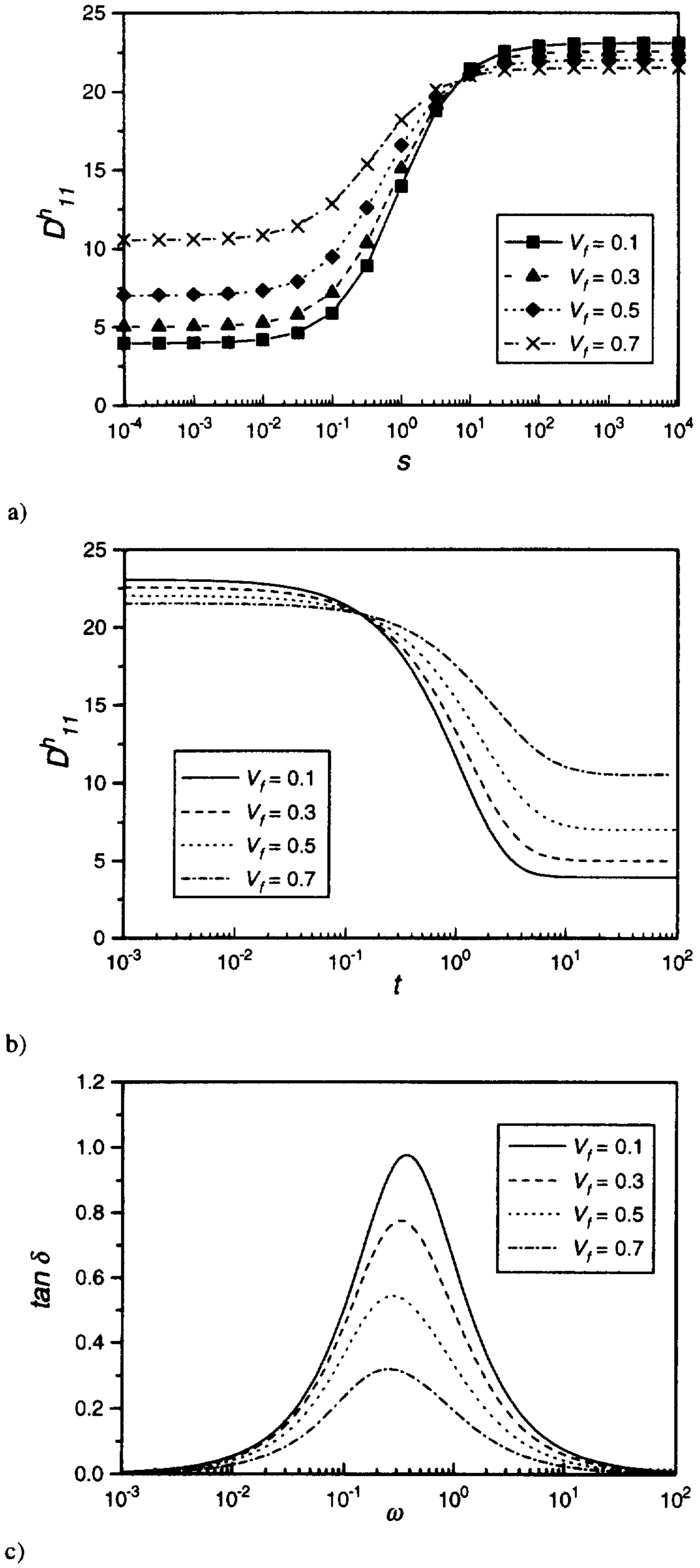


Fig. 5. Effect of volume fraction : (a) effective modulus in Laplace transformed domain ; (b) effective modulus in time domain ; and (c) effective loss tangent in frequency domain.

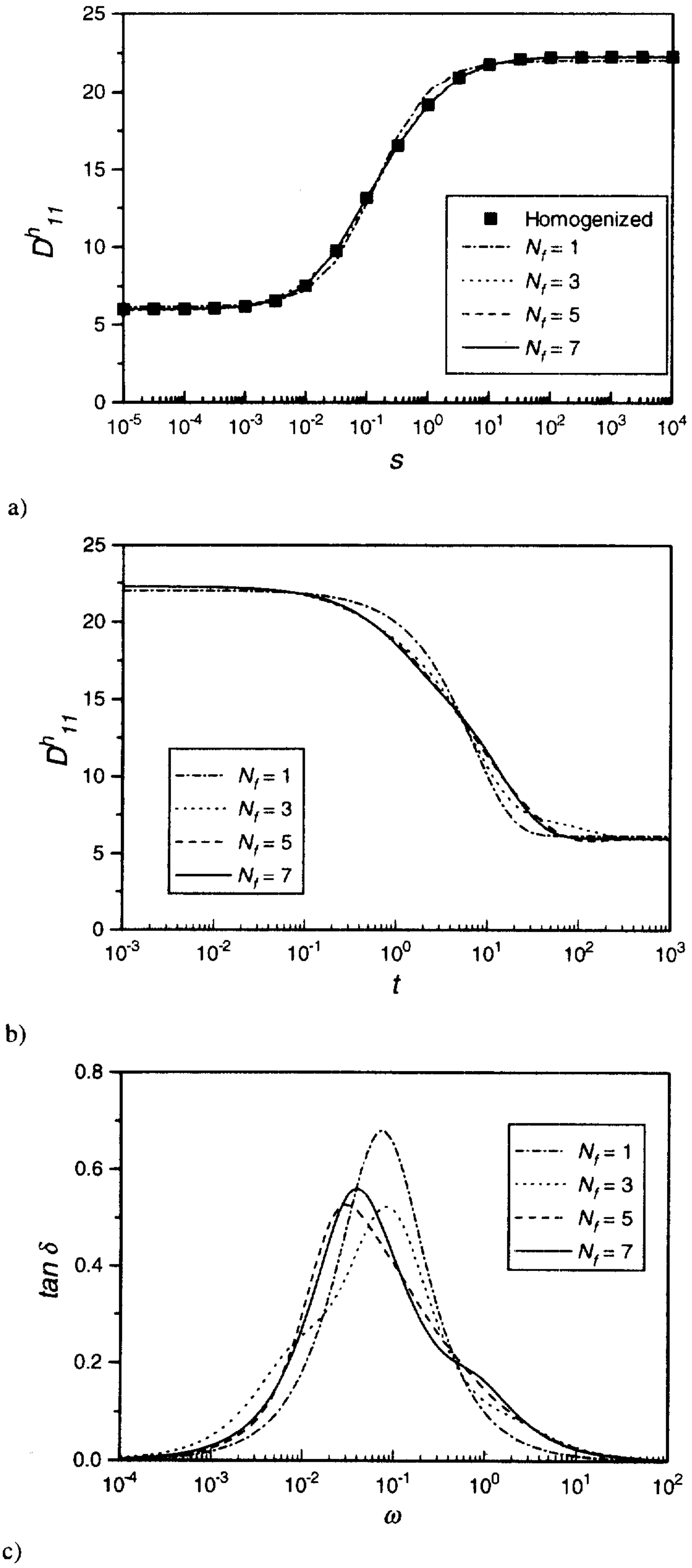
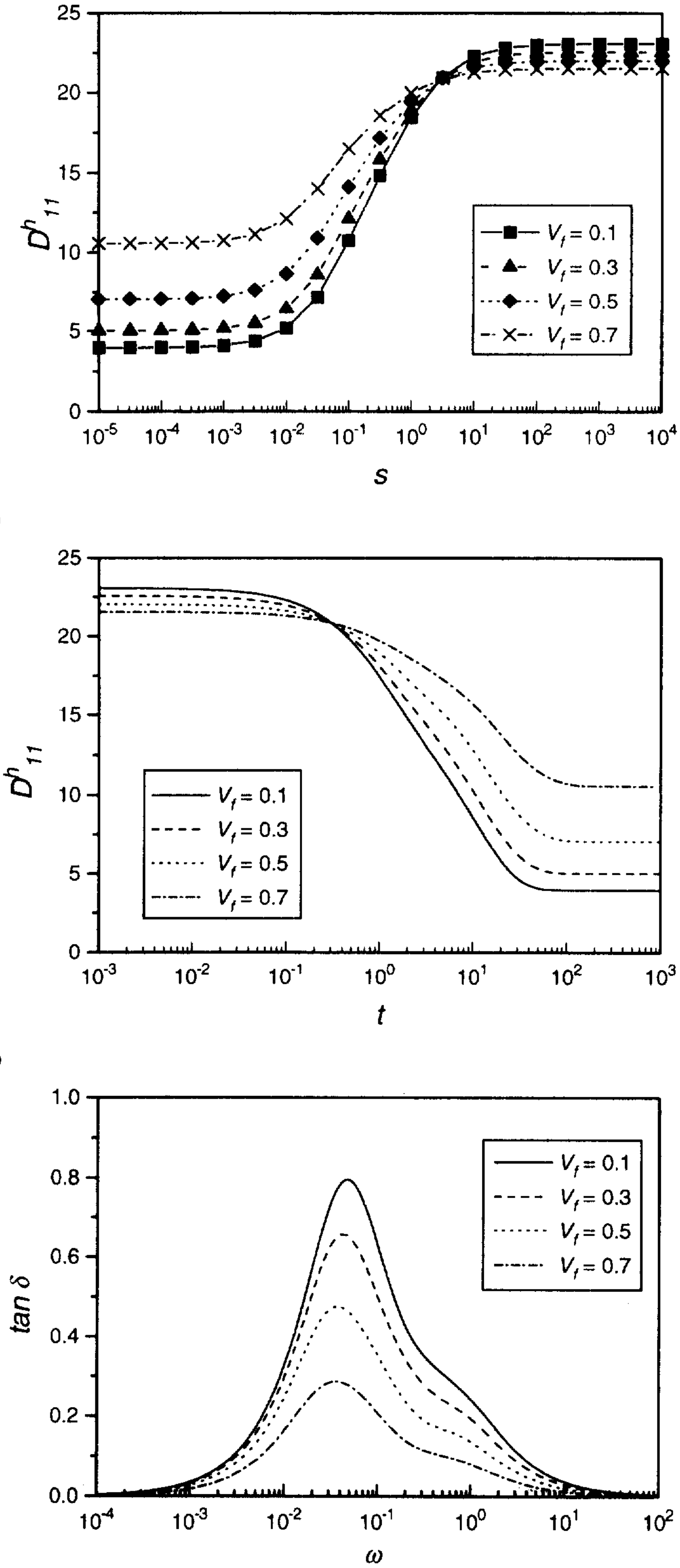


Fig. 6. Effect of number of fitting terms: (a) effective modulus in Laplace transformed domain; (b) effective modulus in time domain; and (c) effective loss tangent in frequency domain.



a)

b)

c)

Fig. 7. Effect of volume fraction : (a) effective modulus in Laplace transformed domain ; (b) effective modulus in time domain ; and (c) effective loss tangent in frequency domain.

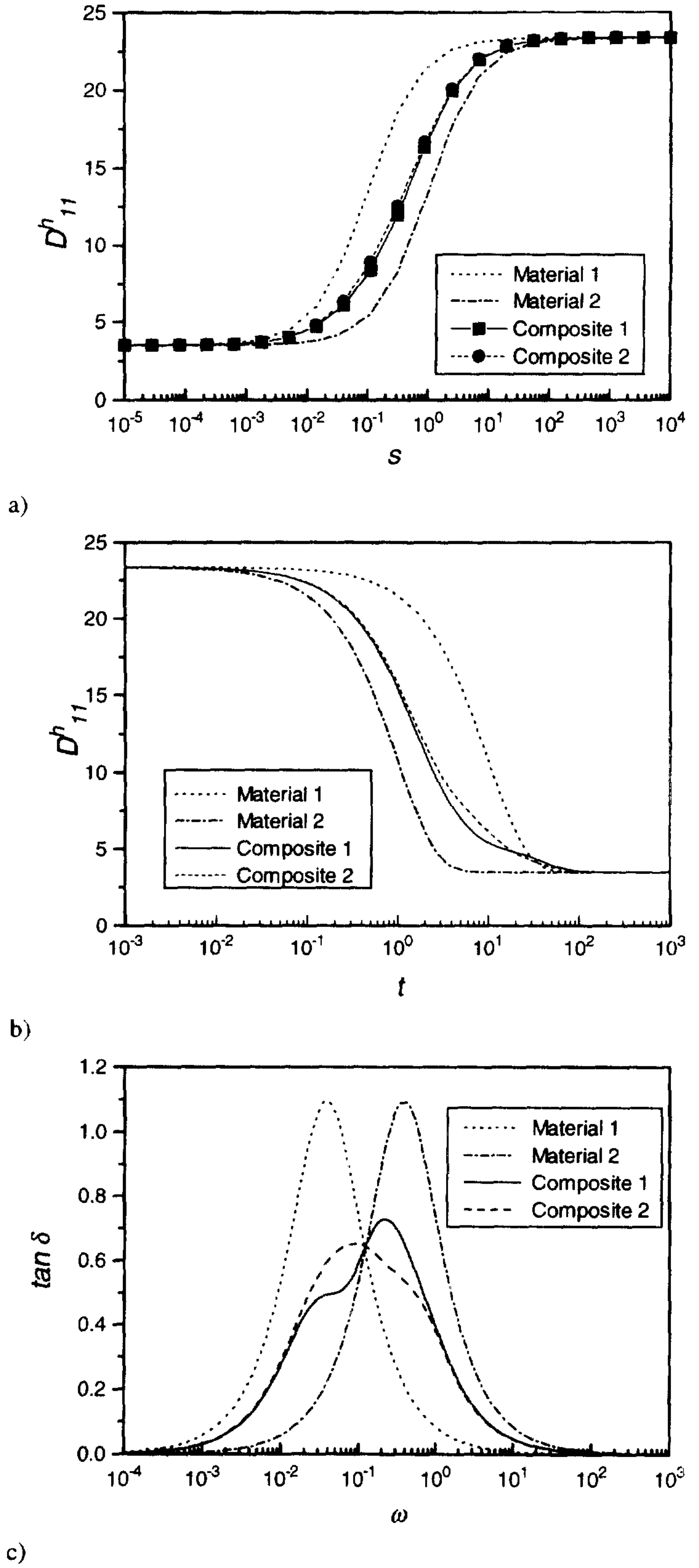


Fig. 8. Composite of two viscoelastic materials: (a) effective modulus in Laplace transformed domain; (b) effective modulus in time domain; and (c) effective loss tangent in frequency domain.

of material 1 and 2 are interchanged. The volume fraction of material 1 and 2 are the same in both composites, i.e. 50%, respectively. Figure 8 shows the computed results. The loss tangents and thus the damping effects of the homogenized materials as a function of frequency may be significantly affected by the configurations of the microstructure as well as the relaxation moduli of the constituent materials. The loss tangent at a fixed frequency seems to be bounded by the largest loss tangent of the constituent materials at that frequency. However, suppose the operating frequency is not fixed at one frequency, but varied at several frequencies or in some frequency interval. In those cases, even when the dampings of the constituents themselves are low, it may be possible to fabricate a composite, with a special microstructural configuration, that has high damping at those frequencies or in that frequency interval.

5. CONCLUSIONS

A method for estimating the homogenized relaxation moduli of the general linear viscoelastic composite materials has been presented. The memory effects have been presented in general form and it has been shown that the memory effects disappear if the relaxation moduli are separable in space and time. For the typical fiber reinforced composites, where the elastic fibers are much stiffer than the polymer matrices, the memory effects become negligible. In the cases where the memory effects do not appear, it is sufficient to compute the relaxation moduli for only one s in Laplace transformed domain and inverse Laplace transform is readily obtained since the shape of the graph of the relaxation function against time is not changed. However, for the composites where the constituent materials have comparable stiffnesses, the memory effects may become apparent. The memory effects make numerical inverse Laplace transform very complicated.

The memory effects and inverse Laplace transform have been discussed in detail. Due to the presence of the memory effects, additional terms are required in the Prony series approximation of the homogenized relaxation moduli for accurate inverse Laplace transform. It also has been shown that maximum damping can be achieved by choosing a right configuration of the microstructure.

Designing an optimal microstructure of viscoelastic composites with required damping capacity or the prescribed effective moduli could be our future work.

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